Continuum Hypothesis
Composer's Note, Dec 2020
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It's not often that someone asks me to write a math-related song, so when Mark Weiss of Earthwise Productions/Lions Without Wings contacted me about composing a musical response to the Continuum Hypothesis, I was thrilled. Math has been an ongoing part of my life, and I sometimes look for ways to combine math with my music. My album *Prime Knot* was an instrumental take on this, *Asymptote* is a song that uses a math metaphor as the lyrics. Sometimes I write lyric 'parodies' of famous songs with math concepts. But this is the first song I've written about a new topic I recently learned about.

As an undergraduate math major at Columbia, I didn't learn about the Continuum Hypothesis (CH). At least I don't remember that I did! It was a challenge to learn about this complex issue after having forgotten so much math-beyond-calculus, and even harder to get a handle of Paul Cohen's forcing technique. Here's an attempt at a basic summary, for those reading without any math background.

**Skip to the end if you just want to read my 'take' on things *

CH: There is no set whose cardinality is strictly between that of the integers and the real numbers.
The truth of this statement was a central question in mathematics in the 20th century, a time when many mathematicians were obsessed with the idea of the infinite.
Of the many different sizes of infinity [https://en.wikipedia.org/wiki/Infinity], the ones you are probably most familiar with are the set of integers (…-2, -1, 0, 1, 2, 3…) and the set of real numbers. Each set is infinite, but it has a "size", or "cardinality". If you look at how large each set is, the CH says there's no set that has a cardinality (size) between the two. The "continuum" refers to the real numbers.

Georg Cantor asserted this hypothesis in 1878, saying that any infinite subset of the reals can be put into a 1-1 correspondence (bijection) with the set of integers or the set of reals.
David Hilbert asked the question of whether this was true in 1900; it's the first in his famous list of 23 Problems [https://en.wikipedia.org/wiki/Hilbert%27s_problems].
In 1922, Skolem speculated that the truth of the CH was independent of the axioms for set theory given by Zermelo in 1908 (the standard set theory created at the time).
In 1937, Kurt Gödel proved that CH is consistent with the axioms of standard set theory, and that you can't prove the set-of-intermediate size exists (using standard set theory axioms). This showed independence of the first "half" of the CH.
[https://plato.stanford.edu/entries/goedel/#GodWorSetThe]

Building off of this, in 1961, Paul Cohen proved that the opposite, or negation, of CH (~CH) is *also* consistent with the axioms of standard set theory. This, together with Gödel's result, means that the truth of CH is independent of ZF set theory (with or without the Axiom of Choice). Cohen proved that you can't prove the intermediate-size-set *doesn't* exist.

Therefore, the existence of intermediate-size-set, (with a size between aleph 0(integers) and aleph 1(reals)), is not provable. Gödel showed CH *cannot be disproved* even if the axiom of choice (AC) is adopted. Cohen showed that CH *cannot be proven* from axioms of ZFC (standard set theory with axiom of choice), which completed the overall independence of CH proof. CH's independence from ZFC means that its truth value (proving it's true or not true) is impossible using regular set theory. https://en.wikipedia.org/wiki/Continuum_hypothesis

Paul Cohen did this using a technique he developed in 1961 called "forcing", which is now a standard technique used in many other areas of set theory, analysis, mathematics, and logic. He won the Fields Medal in 1963 for this work.

The forcing technique involves proving things are true in stages.
Starting with a model where CH works, you begin to construct another model with more sets than the original. These additional sets are constructed in such a way that a contradiction unfolds (ie CH fails). So that means that the truth of the CH is independent of the usual set theory people used at the time (ZF). Two brief descriptions of forcing: "The method begins with a model of ZF in which CH holds, and constructs another model which contains more sets than the original, in a way that CH does not hold in the new model" (https://en.wikipedia.org/wiki/Continuum_hypothesis#Independence_from_ZFC) or, as explained by Solomon Feferman, a famous logician of Stanford University, in this article about Cohen (https://news.stanford.edu/news/2007/april4/cohen-040407.html), "Forcing is used to construct unusual models of the axioms of set theory, whereby statements are made true in stages and forced to remain so in all further stages."

In this Video of Paul Cohen's talk at Gödel Centennial (Vienna, 2006), https://youtu.be/VBFLWk7k1Zo, Cohen talks about how he developed the forcing technique, how the idea came to him, and how excited he was to share it with Gödel at Princeton. He makes it sound incredibly "easy", simple, and beautiful.

Some people were unsatisfied with the independence or "inconclusive" results. Pluralists (like Cohen), said that the independence results settled the question of CH, showing it had no answer. Non-pluralists, like Gödel, thought that the independence result showed a weakness in set theory, and that new axioms needed to be created (large cardinal axioms). https://plato.stanford.edu/entries/continuum-hypothesis/ My understanding is that the truth of the CH depends on what universe you're working in in, and how you want to look at things. This is as much a question of set theory and logic, as it is a mathematical theory.

This stuff is very complex and hard to understand; a graduate student in math or logic with more of a background in set theory would have a better handle on it than me. Still I tried to absorb what I could. At the bottom is a list of some intro articles on the topic, though you may need some set theory or math background.

If you skipped the math:
The widely-accepted view on CH is that its truth is independent of standard set theory, and that ***its truth is undecidable*. So it can't be proven true or false, using standard set theory. Or, it can be true or not true, depending on how you look at it. That has different implications for

what you want to do. AND, we don't know whether an infinite set that's between the size of the reals and integers exists – you can't prove that it exists, but you also can't prove it doesn't exist. So we *can't* know. To most people, that settles the question; the resolution is that it's undecidable, that there is no answer. That's out there.

People often think that math is black-and-white, yes-or-no, and that things are clear-cut, orderly, and provable. They also often think of math as being absolute, and don't realize that mathematicians are exploring new ideas and trying things out all the time, that there are new discoveries being made. Sometimes there are partially-answered questions or results that are not yet conclusive, which may take generations of mathematicians to advance . Learning about CH took me on a journey of also looking at the lives of the mathematicians who advanced these theories, their thought processes, and the implications of their work in set theory, math, and beyond.

At its highest levels, advanced mathematics and logic requires a lot of creativity – just like in art.

This song is my response to all of these ideas. It is dedicated to Paul Cohen, 1934-2007, one of the greatest mathematicians of the 20th century.

Thanks to Jeff Helzner, Alex Kontorovich at the National Museum of Mathematic's *Ask a Mathematician -- Anything!* series, and Alex Moll for pointing me towards some great introductory resources and for talking to me about CH.

Video of Paul Cohen's talk at Gödel Centennial, Vienna, 2006
https://youtu.be/VBFLWk7k1Zo

Stanford Encyclopedia of Philosophy:
https://plato.stanford.edu/entries/continuum-hypothesis/

Intro article on infinities
https://www.quantamagazine.org/mathematicians-measure-infinities-find-theyre-equal-20170912/

Wikipedia on CH
https://en.wikipedia.org/wiki/Continuum_hypothesis

Introduction to Forcing
https://arxiv.org/abs/0712.2279